Ergodic Theory
Problem Sheet 13

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P1. Let $(X, \mathscr{B}, \mu)$ be a probability space, $\mathscr{A} \subseteq \mathscr{B}$ be a σ-algebra and $f \in L^1(X, \mathscr{B}, \mu)$ be a real-valued function. Let $m = \inf_{y \in X} f(y)$ and $M = \sup_{y \in X} f(y)$. Show that $\mathbb{E}(f \mid \mathscr{A})(x) \in [m, M]$ for almost every $x \in X$.

P2. Let $(X, \mathscr{B}, \mu)$ be a probability space, $\mathscr{A} \subseteq \mathscr{B}$ be a σ-algebra and $f \in L^1(X, \mathscr{B}, \mu)$. Show that

$$|\mathbb{E}(f \mid \mathscr{A})(x)| \leq \mathbb{E}(|f| \mid \mathscr{A})(x)$$

for almost every $x \in X$.

P3. Let $(X, \mathscr{A}, \mu)$ be a probability space, $\mathscr{B}$ a sub-σ-algebra of $\mathscr{A}$, and $x \to \mu_x$ the disintegration of μ with respect $\mathscr{B}$.

(a) Prove that for all $A \in \mathscr{B}$, $\mu_x(A) = \mathbb{1}_A(x)$, for μ-almost every $x \in X$.

(b) Prove that the measures μ_x are constant on almost every fiber, namely that for μ-almost every $x \in X$,

$$\mu_y = \mu_x, \quad \text{for } \mu_x\text{-almost every } y \in X.$$

(c) Now suppose that $(X, \mathscr{A}, \mu, T)$ is an invertible measure-preserving system and that $\mathscr{B}$ is a T-invariant sub-σ-algebra. Prove that

$$T\mu_x = \mu_{Tx},$$

for μ-almost every $x \in X$.